

On the Strengths of Connectivity and Robustness in General Random Intersection Graphs

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Abstract — Random intersection graphs have received much attention for nearly two decades, and currently have a wide range of applications ranging from key predistribution in wireless sensor networks to modeling social networks. In this paper, we investigate the strengths of connectivity and robustness in a *general random intersection graph* model. Specifically, we establish sharp asymptotic zero–one laws for k -connectivity and k -robustness, as well as the asymptotically exact probability of k -connectivity, for any positive integer k . The k -connectivity property quantifies how resilient is the connectivity of a graph against node or edge failures. On the other hand, k -robustness measures the effectiveness of *local* diffusion strategies (that do not use global graph topology information) in spreading information over the graph in the presence of misbehaving nodes. In addition to presenting the results under the general random intersection graph model, we consider two special cases of the general model, a *binomial* random intersection graph and a *uniform* random intersection graph, which both have numerous applications as well. For these two specialized graphs, our results on asymptotically exact probabilities of k -connectivity and asymptotic zero–one laws for k -robustness are also novel in the literature.

Index Terms— Connectivity, consensus, random graph, random intersection graph, random key graph, robustness.

I. INTRODUCTION

A. Graph Models

Random intersection graphs have been introduced by Singer-Cohen [1] and received considerable attention [2]–[17] for nearly two decades. In these graphs, each node is assigned a set of *objects* selected by some random mechanism. An undirected edge exists between any two nodes that have at least one object in common. Random intersection graphs have proved useful in modeling and analyzing real-world networks in a wide variety of application areas. Examples include secure wireless sensor networks [2]–[7], frequency hopping spread spectrum [3], spread of epidemics [8], [10], and social and information networks [7]–[9] including collaboration networks [8], [9] and common-interest networks [7]. Several classes of random intersection graphs have been analyzed, and results concerning various graph properties such as clustering [9], component evolution [2], [11] and degree distribution [12] have been obtained.

The model considered in this paper, hereafter referred to as a *general random intersection graph*, represents a generalization [2], [9], [12] of random intersection graphs. It

is defined on a node set $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ as follows. Each node v_i ($i = 1, 2, \dots, n$) is assigned an object set S_i from an object pool \mathcal{P} consisting of P_n distinct objects, where P_n is a function of n . Each object S_i is constructed using the following two-step procedure: First, the size of S_i , $|S_i|$, is determined according to some probability distribution $\mathcal{D} : \{1, 2, \dots, P_n\} \rightarrow [0, 1]$. Of course, we have $\sum_{x=1}^{P_n} \mathbb{P}[|S_i| = x] = 1$, with $\mathbb{P}[A]$ denoting the probability that event A occurs. Next, S_i is formed by selecting $|S_i|$ distinct objects uniformly at random from the object pool \mathcal{P} . In other words, conditioning on $|S_i| = s_i$, set S_i is chosen uniformly among all s_i -size subsets of \mathcal{P} . This process is repeated independently for all object sets S_1, \dots, S_n . Finally, an undirected edge is assigned between two nodes if and only if their corresponding object sets have at least one object in common; namely, distinct nodes v_i and v_j have an edge in between if and only if $S_i \cap S_j \neq \emptyset$. The graph defined through this adjacency notion is denoted by $G(n, P_n, \mathcal{D})$.

A specific case of the general model $G(n, P_n, \mathcal{D})$, known as the *binomial* random intersection graph, has been widely explored to date [9]–[14]. Under this model, each object set S_i is constructed by a Bernoulli-like mechanism; i.e., by adding each object to S_i independently with probability p_n . Like integer P_n , probability p_n is also a function of n . The term “binomial” accounts for the fact that $|S_i|$ now follows a binomial distribution with P_n as the number of trials and p_n as the success probability in each trial. We denote the binomial random intersection graph by $G_b(n, P_n, p_n)$, where subscript “b” stands for “binomial”.

Another well-known special case of the general model $G(n, P_n, \mathcal{D})$ is the *uniform* random intersection graph [4]–[6], [15]–[17]. Under the uniform model, the probability distribution \mathcal{D} concentrates on a single integer K_n , where $1 \leq K_n \leq P_n$; i.e., for each node v_i , the object set size $|S_i|$ equals K_n with probability 1. P_n and K_n are both integer functions of n . We denote by $G_u(n, P_n, K_n)$ the uniform random intersection graph, with “u” meaning “uniform”.

A concrete example for the application of random intersection graphs can be given in the context of secure wireless sensor networks. As explained in detail in numerous other places [2]–[5], [7], [8], [10], [11], the uniform random intersection graph model $G_u(n, P_n, K_n)$ is induced naturally by the Eschenauer–Gligor (EG) random key predistribution

scheme [6], which is a typical solution to ensure secure communications in wireless sensor networks. In particular, let the set of n nodes in graph $G_u(n, P_n, K_n)$ stand for the n sensors in the wireless network. Also, let the object pool \mathcal{P} (with size P_n) represent the set of cryptographic keys available to the network and let K_n be the number of keys assigned to each sensor (selected uniformly at random from the key pool \mathcal{P}). Then, the edges in $G_u(n, P_n, K_n)$ represent pairs of sensors that share at least one cryptographic key and thus that can *securely* communicate over existing wireless links in the EG scheme. In the above application, objects that nodes have are cryptographic keys, so uniform random intersection graphs are also referred to as random key graphs [3], [4], [17].

In the secure sensor network area, the general random intersection graph model in this paper captures the differences that may exist among the number of keys possessed by each sensor. This may occur for various reasons that include (a) the assigned numbers of keys on sensors may vary prior to deployment given the heterogeneity in available sensor memory [2]; (b) the number of keys available to a sensor may decrease after deployment due to revocation of compromised keys [7]; and (c) the number of keys on a sensor may increase due to the path key establishment phase of the EG scheme [6], where new path keys are generated and distributed to participating sensors.

B. (k -)Connectivity and (k -)Robustness

We now introduce the graph properties that we are interested in. First, a graph is connected if there exists at least a path of edges between any two nodes [18]. A graph is said to be k -connected if each pair of nodes has at least k internally node-disjoint path(s) in between [14]; equivalently, a graph is k -connected if it can not be made disconnected by deleting at most $(k - 1)$ nodes or edges.¹ In this manner, k -connectivity quantifies the resiliency of graph connectivity against node or edge failures. In addition, it enables multi-path routing, and is also useful to achieve consensus in the graph [7]. In particular, to achieve consensus in the presence of m adversarial nodes in a large-scale graph (with node size greater than $3m$), a necessary and sufficient condition is that the graph is $(2m + 1)$ -connected [21].

Many algorithms have been proposed to achieve consensus [27]–[33] in graphs with sufficient connectivity. However, these algorithms typically assume that nodes have full knowledge of the graph topology, which is impractical in some cases [27]. To this end, Zhang and Sundaram [27] introduce the notion of *graph robustness*. They show that when nodes are limited to local information instead of the global graph topology, consensus can be reached in a sufficiently robust graph in the presence of adversarial/misbehaving nodes, but not in a sufficiently connected and insufficiently robust

graph. Therefore, graph robustness quantifies the effectiveness and resiliency of local-information-based consensus algorithms in the presence of adversarial/misbehaving nodes. Robustness is an important property with broad relevance in graph processes beyond consensus; e.g., robustness plays a key role in information cascades and contagion processes [27]. It is worth noting that robustness is a stronger property than connectivity in the sense that any k -robust graph is also k -connected, whereas a k -connected graph is not necessarily k -robust [27].

Formally, a graph with a node set \mathcal{V} is k -robust if at least one of (a) and (b) below hold for any non-empty and strict subset T of \mathcal{V} : (a) there exists at least a node $v_a \in T$ such that v_a has no less than k neighbors inside $\mathcal{V} \setminus T$; and (b) there exists at least a node $v_b \in \mathcal{V} \setminus T$ such that v_b has no less than k neighbors inside T .

C. Contributions and Organization

With various applications of random intersection graphs, and k -connectivity and k -robustness graph properties in mind, a natural question to ask is whether random intersection graphs are k -connected or k -robust under certain conditions? Our paper answers this question. We summarize our contributions as follows:

- i) We derive sharp zero–one laws and asymptotically exact probabilities for k -connectivity in general random intersection graphs.
- ii) We establish sharp zero–one laws for k -robustness in general random intersection graphs.
- iii) For the two specific instances of the general graph model, a binomial random intersection graph and a uniform random intersection graph, we provide the first results on the asymptotically exact probabilities of k -connectivity and zero–one laws for k -robustness.

The rest of the paper is organized as follows. Section II presents the main results as Theorems 1–6. Then, we introduce some auxiliary facts and lemmas in Section III, before establishing the main results in Sections IV and V. Section VI details the proofs of the lemmas. We provide numerical experiments in Section VII. Section VIII reviews related work; and Section IX concludes the paper.

II. THE RESULTS

Our main results are presented in Theorems 1–6 below. We defer the proofs of all theorems to Sections IV and V. Throughout the paper, k is a positive integer and does not scale with n ; and e is the base of the natural logarithm function, \ln . All limits are understood with $n \rightarrow \infty$. We use the standard Landau asymptotic notation $o(\cdot)$, $O(\cdot)$, $\omega(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$ and \sim ; in particular, for two positive functions $f(n)$ and $g(n)$, the relation $f(n) \sim g(n)$ signifies $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$. For a random variable X , the terms $\mathbb{E}[X]$ and $\text{Var}[X]$ stand for its expected value and variance, respectively.

A. Zero–One Laws and Exact Probabilities for Asymptotic k -Connectivity

We provide zero–one laws and exact probabilities for asymptotic k -connectivity in different graphs below.

¹As in much other work [13], [14], [24]–[26], [35], k -connectivity in this paper means k -vertex-connectivity in graph theory [18], [19], [24]. Yet, results on k -edge-connectivity similar to those in Theorems 1–3 of Section II-A are shown to hold as well in the full version [20].

1) *k-Connectivity in General Random Intersection Graphs:* Theorem 1 below presents a zero-one law and the exact probability for asymptotic k -connectivity in a general random intersection graph.

Theorem 1: Consider a general random intersection graph $G(n, P_n, \mathcal{D})$. Let X be a random variable following probability distribution \mathcal{D} . With a sequence α_n for all n defined through

$$\frac{\{\mathbb{E}[X]\}^2}{P_n} = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n}, \quad (1)$$

if $\mathbb{E}[X] = \Omega(\sqrt{\ln n})$, $\text{Var}[X] = o\left(\frac{\{\mathbb{E}[X]\}^2}{n(\ln n)^2}\right)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{Graph } G(n, P_n, \mathcal{D}) \text{ is } k\text{-connected.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty, \\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \alpha^* \in (-\infty, \infty). \end{cases} \quad \square$$

2) *k-Connectivity in Binomial Random Intersection Graphs:* Theorem 2 below presents a zero-one law and the exact probability for asymptotic k -connectivity in a binomial random intersection graph.

Theorem 2: For a binomial random intersection graph $G_b(n, P_n, p_n)$, with a sequence α_n for all n defined through

$$p_n^2 P_n = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n}, \quad (2)$$

if $P_n = \omega(n(\ln n)^5)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ is } k\text{-connected.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty, \\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \alpha^* \in (-\infty, \infty). \end{cases} \quad \square$$

Remark 1: As we will explain in Section IV-B within the proof of Theorem 2, for the zero-one law, the condition $P_n = \omega(n(\ln n)^5)$ can be weakened as $P_n = \Omega(n(\ln n)^5)$, while we enforce $P_n = \omega(n(\ln n)^5)$ for the asymptotically exact probability result. \square

3) *k-Connectivity in Uniform Random Intersection Graphs:* Theorem 3 below presents a zero-one law and the exact probability for asymptotic k -connectivity in a uniform random intersection graph.

Theorem 3: For a uniform random intersection graph $G_u(n, P_n, K_n)$, with a sequence α_n for all n defined through

$$\frac{K_n^2}{P_n} = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n}, \quad (3)$$

if $K_n = \Omega(\sqrt{\ln n})$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{Graph } G_u(n, P_n, K_n) \text{ is } k\text{-connected.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty, \\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \alpha^* \in (-\infty, \infty). \end{cases} \quad \square$$

B. *Zero-One Laws for Asymptotic k-Robustness*

We provide zero-one laws for asymptotic k -robustness in different graphs below.

1) *k-Robustness in General Random Intersection Graphs:* Theorem 4 as follows gives a zero-one law for asymptotic k -robustness in a general random intersection graph.

Theorem 4: Consider a general random intersection graph $G(n, P_n, \mathcal{D})$. Let X be a random variable following probability distribution \mathcal{D} . With a sequence α_n for all n defined through

$$\frac{\{\mathbb{E}[X]\}^2}{P_n} = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n}, \quad (4)$$

if $\mathbb{E}[X] = \Omega((\ln n)^3)$, $\text{Var}[X] = o\left(\frac{\{\mathbb{E}[X]\}^2}{n(\ln n)^2}\right)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{Graph } G(n, P_n, \mathcal{D}) \text{ is } k\text{-robust.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty. \end{cases} \quad \square$$

2) *k-Robustness in Binomial Random Intersection Graphs:* Theorem 5 below gives a zero-one law for asymptotic k -robustness in a binomial random intersection graph.

Theorem 5: For a binomial random intersection graph $G_b(n, P_n, p_n)$, with a sequence α_n for all n defined through

$$p_n^2 P_n = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n}, \quad (5)$$

if $P_n = \Omega(n(\ln n)^5)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ is } k\text{-robust.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty. \end{cases} \quad \square$$

3) *k-Robustness in Uniform Random Intersection Graphs:* Theorem 6 below gives a zero-one law for asymptotic k -robustness in a uniform random intersection graph.

Theorem 6: For a uniform random intersection graph $G_u(n, P_n, K_n)$, with a sequence α_n for all n defined through

$$\frac{K_n^2}{P_n} = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n}, \quad (6)$$

if $K_n = \Omega((\ln n)^3)$ and $|\alpha_n| = o(\ln n)$, then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{Graph } G_u(n, P_n, K_n) \text{ is } k\text{-robust.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty. \end{cases} \quad \square$$

In view of Theorems 1–6, for each general/binomial/uniform random intersection graph, its k -connectivity and k -robustness asymptotically obey the same zero-one laws. Moreover, these zero-one laws are all *sharp* since $|\alpha_n|$ can be much smaller compared to $\ln n$; e.g., even $\alpha_n = \pm c \cdot \ln \ln \dots \ln n$ with an arbitrary positive constant c satisfies $\lim_{n \rightarrow \infty} \alpha_n = \pm \infty$.

III. AUXILIARY FACTS AND LEMMAS

We present a few facts and lemmas which are used to establish the theorems. To begin with, recalling that k does not scale with n , we obtain Facts 1 and 2 below, whose proofs are straightforward and thus omitted here.

Fact 1: For $|\alpha_n| = o(\ln n)$, it holds that

$$\frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n} \sim \frac{\ln n}{n}.$$

Fact 2: For $|\alpha_n| = o(\ln n)$, we have

$$\begin{aligned} & \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n} \cdot \left[1 \pm o\left(\frac{1}{\ln n}\right) \right] \\ &= \frac{\ln n + (k-1) \ln \ln n + \alpha_n \pm o(1)}{n}, \end{aligned}$$

and

$$\begin{aligned} & \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n} \cdot \left[1 \pm O\left(\frac{1}{\ln n}\right) \right] \\ &= \frac{\ln n + (k-1) \ln \ln n + \alpha_n \pm O(1)}{n}. \end{aligned}$$

Lemma 1 below presents the result on k -robustness of an Erdős-Rényi graph. An Erdős-Rényi graph $G(n, \hat{p}_n)$ [18] is defined on a set of n nodes such that any two nodes have an edge in between independently with probability \hat{p}_n .

Lemma 1: For an Erdős-Rényi graph $G(n, \hat{p}_n)$, with a sequence α_n for all n through

$$\hat{p}_n = \frac{\ln n + (k-1) \ln \ln n + \alpha_n}{n}, \quad (7)$$

then it holds that

$$\lim_{n \rightarrow \infty} \mathbb{P}[G(n, \hat{p}_n) \text{ is } k\text{-robust.}] = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty. \end{cases} \quad (8)$$

To prove Lemma 1, we note the following three facts. (a) The desired result (8) with $|\alpha_n| = o(\ln \ln n)$ is demonstrated in [27, Theorem 3]. (b) By [14, Facts 3 and 7], for any monotone increasing graph property \mathcal{I} , the probability that graph $G(n, \hat{p}_n)$ has property \mathcal{I} is non-decreasing as \hat{p}_n increases. (c) k -robustness is a monotone increasing graph property according to [34, Lemma 3]. In view of (a) (b) and (c) above, we obtain Lemma 1.

Throughout Lemmas 2–5 below, \mathcal{I} is an arbitrary monotone increasing graph property, where a graph property is called monotone increasing if it holds under the addition of edges. Except Lemma 4 which is from [2, Lemma 4], the proofs of Lemmas 2, 3 and 5 are deferred to Section VI.

Lemma 2: Let X be a random variable with probability distribution \mathcal{D} . If $\text{Var}[X] = o\left(\frac{\{\mathbb{E}[X]\}^2}{n(\ln n)^2}\right)$, then there exists $\epsilon_n = o\left(\frac{1}{\ln n}\right)$ such that

$$\begin{aligned} & \mathbb{P}[\text{Graph } G(n, P_n, \mathcal{D}) \text{ has } \mathcal{I}.] \\ & \geq \mathbb{P}[\text{Graph } G_u(n, P_n, (1 - \epsilon_n)\mathbb{E}[X]) \text{ has } \mathcal{I}.] - o(1), \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}[\text{Graph } G(n, P_n, \mathcal{D}) \text{ has } \mathcal{I}.] \\ & \leq \mathbb{P}[\text{Graph } G_u(n, P_n, (1 + \epsilon_n)\mathbb{E}[X]) \text{ has } \mathcal{I}.] + o(1). \end{aligned} \quad \square$$

Lemma 3: If $p_n = O\left(\frac{1}{n \ln n}\right)$ and $p_n^2 P_n = O\left(\frac{1}{\ln n}\right)$, then there exists $\hat{p}_n = p_n^2 P_n \cdot [1 - O\left(\frac{1}{\ln n}\right)]$ such that

$$\begin{aligned} & \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ has } \mathcal{I}.] \\ & \geq \mathbb{P}[\text{Graph } G(n, \hat{p}_n) \text{ has } \mathcal{I}.] - o(1). \end{aligned} \quad (9)$$

Lemma 4 ([2, Lemma 4]): If $p_n P_n = \omega(\ln n)$, and for all n sufficiently large,

$$\begin{aligned} K_{n,-} & \leq p_n P_n - \sqrt{3(p_n P_n + \ln n) \ln n}, \\ K_{n,+} & \geq p_n P_n + \sqrt{3(p_n P_n + \ln n) \ln n}, \end{aligned}$$

then

$$\begin{aligned} & \mathbb{P}[\text{Graph } G_u(n, P_n, K_{n,-}) \text{ has } \mathcal{I}.] - o(1) \\ & \leq \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ has } \mathcal{I}.] \\ & \leq \mathbb{P}[\text{Graph } G_u(n, P_n, K_{n,+}) \text{ has } \mathcal{I}.] + o(1). \end{aligned}$$

Lemma 5: If $K_n = \omega(\ln n)$ and $p_n = \frac{K_n}{P_n} \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right)$, then

$$\begin{aligned} & \mathbb{P}[\text{Graph } G_u(n, P_n, K_n) \text{ has } \mathcal{I}.] \\ & \geq \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ has } \mathcal{I}.] - o(1). \end{aligned} \quad \square$$

Figure 1 on the next page illustrates the steps of using the lemmas to prove the theorems. Note that the facts used in deriving the theorems are not shown in the plot for brevity.

IV. ESTABLISHING THEOREMS 1–3

Theorems 1–3 describe results on k -connectivity for various random intersection graphs.

A. The Proof of Theorem 1

We demonstrate Theorem 1 with the help of Theorem 3, the proof of which is detailed in Section IV-C.

For any $\epsilon_n = o\left(\frac{1}{\ln n}\right)$, it is clear that

$$(1 \pm \epsilon_n)^2 = 1 \pm 2\epsilon_n + \epsilon_n^2 = 1 \pm o\left(\frac{1}{\ln n}\right). \quad (10)$$

We recall conditions (1) and $|\alpha_n| = o(\ln n)$, which together with (10) and Fact 2 yields

$$\frac{\{(1 \pm \epsilon_n)\mathbb{E}[X]\}^2}{P_n} = \frac{\ln n + (k-1) \ln \ln n + \alpha_n \pm o(1)}{n}. \quad (11)$$

With $\mathbb{E}[X] = \Omega(\sqrt{\ln n})$ and $\epsilon_n = o\left(\frac{1}{\ln n}\right)$, it follows that $(1 \pm \epsilon_n)\mathbb{E}[X] = \Omega(\sqrt{\ln n})$, which along with (11) and $|\alpha_n| = o(\ln n)$ enables the use of Theorem 3 to derive

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}[G_u(n, P_n, (1 \pm \epsilon_n)\mathbb{E}[X]) \text{ is } k\text{-connected.}] \\ & = \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty, \\ e^{-\frac{e^{-\alpha^*}}{(k-1)!}}, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \alpha^* \in (-\infty, \infty). \end{cases} \end{aligned} \quad (12)$$

Since k -connectivity is a monotone increasing graph property [14], Theorem 1 is proved by (12) and Lemma 2. \blacksquare

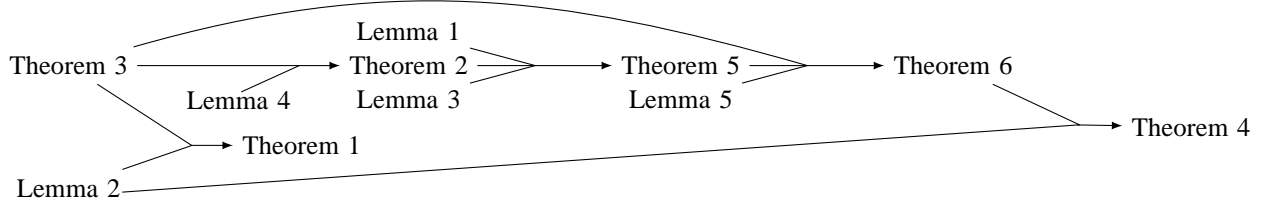


Fig. 1: A plot illustrating the steps of deriving the theorems from the lemmas, with the arrows indicating the directions. For example, Theorem 3 and Lemma 2 are used to prove Theorem 1.

B. The Proof of Theorem 2

From Lemma 4 and Theorem 3, the proof of Theorem 2 is completed once we show that with $K_{n,\pm}$ defined by

$$K_{n,\pm} = p_n P_n \pm \sqrt{3(p_n P_n + \ln n) \ln n}, \quad (13)$$

under conditions of Theorem 2, we have $K_{n,\pm} = \Omega(\sqrt{\ln n})$ and with $\alpha_{n,\pm}$ defined by

$$\frac{K_{n,\pm}^2}{P_n} = \frac{\ln n + (k-1) \ln \ln n + \alpha_{n,\pm}}{n}, \quad (14)$$

then

$$\alpha_{n,\pm} = \alpha_n \pm o(1). \quad (15)$$

From conditions (2) and $|\alpha_n| = o(\ln n)$, and Fact 1, it is clear that

$$p_n^2 P_n \sim \frac{\ln n}{n}. \quad (16)$$

Substituting (16) and condition $P_n = \omega(n(\ln n)^5)$ into (13), it holds that

$$K_{n,\pm} = \omega((\ln n)^3) = \Omega(\sqrt{\ln n}), \quad (17)$$

and

$$\frac{K_{n,\pm}^2}{P_n} = p_n^2 P_n \cdot \left[1 \pm o\left(\frac{1}{\ln n}\right) \right]. \quad (18)$$

Then from (2) (14) (18) and Fact 2, we obtain (15). As explained before, with (14) (15) and (17), Theorem 2 is proved from Lemma 4 and Theorem 3. ■

As noted in Remark 1, to prove only the zero-one law but not the asymptotically exact probability result in Theorem 2, condition $P_n = \omega(n(\ln n)^5)$ can be weakened as $P_n = \Omega(n(\ln n)^5)$. This can be seen by the argument that under $P_n = \Omega(n(\ln n)^5)$, (15) can be weakened as $\alpha_{n,\pm} = \alpha_n \pm O(1)$, which can still be used to establish the zero-one law.

C. The Proof of Theorem 3

We derive in [35] the asymptotically exact probability and an asymptotic zero-one law for k -connectivity in graph $G(n, \tilde{p}_n) \cap G_u(n, P_n, K_n)$, which is the superposition of an Erdős-Rényi graph $G(n, \tilde{p}_n)$ on a uniform random intersection graph $G_u(n, P_n, K_n)$. Setting $\tilde{p}_n = 1$, graph $G(n, \tilde{p}_n) \cap G_u(n, P_n, K_n)$ becomes $G_u(n, P_n, K_n)$. Then with $\tilde{p}_n = 1$, we obtain from [35, Theorem 1] that if $P_n = \Omega(n)$ and

$$1 - \left(\frac{P_n - K_n}{K_n} \right) / \left(\frac{P_n}{K_n} \right) = \frac{\ln n + (k-1) \ln \ln n + \beta_n}{n}, \quad (19)$$

then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}[G_u(n, P_n, K_n) \text{ is } k\text{-connected.}] \\ &= \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \beta_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \beta_n = \infty, \\ e^{-\frac{e^{-\beta^*}}{(k-1)!}}, & \text{if } \lim_{n \rightarrow \infty} \beta_n = \beta^* \in (-\infty, \infty). \end{cases} \end{aligned} \quad (20)$$

Note that if $\beta_n = \alpha_n \pm o(1)$, then (i) $\lim_{n \rightarrow \infty} \beta_n$ exists if and only if $\lim_{n \rightarrow \infty} \alpha_n$ exists; and (ii) when they both exist, $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \alpha_n$. Therefore, Theorem 3 is proved once we show $P_n = \Omega(n)$ and (19) with $\beta_n = \alpha_n \pm o(1)$ given conditions $K_n = \Omega(\sqrt{\ln n})$, $|\alpha_n| = o(\ln n)$ and (3).

From $|\alpha_n| = o(\ln n)$, (3) and Fact 1, it holds that

$$\frac{K_n^2}{P_n} \sim \frac{\ln n}{n}, \quad (21)$$

which along with $K_n = \Omega(\sqrt{\ln n})$ yields

$$P_n \sim \frac{n K_n^2}{\ln n} = \Omega(n).$$

We derive in [7, Lemma 8] that

$$1 - \left(\frac{P_n - K_n}{K_n} \right) / \left(\frac{P_n}{K_n} \right) = \frac{K_n^2}{P_n} \cdot \left[1 \pm O\left(\frac{K_n^2}{P_n}\right) \right]. \quad (22)$$

Applying (21) to (22),

$$1 - \left(\frac{P_n - K_n}{K_n} \right) / \left(\frac{P_n}{K_n} \right) = \frac{K_n^2}{P_n} \cdot \left[1 \pm o\left(\frac{1}{\ln n}\right) \right],$$

which together with (3) and Fact 2 leads to (19) with condition $\beta_n = \alpha_n \pm o(1)$. Since we have proved $P_n = \Omega(n)$ and (19) with $\beta_n = \alpha_n \pm o(1)$, Theorem 3 follows from (20). ■

V. ESTABLISHING THEOREMS 4–6

Theorems 4–6 present results on k -robustness for various random intersection graphs.

A. The Proof of Theorem 4

Similar to the process of proving Theorem 1 with the help of Theorem 3, we demonstrate Theorem 4 using Theorem 6, the proof of which is given in Section V-C.

Note that condition (4) is the same as (1), and condition $|\alpha_n| = o(\ln n)$ holds. Then as shown in Theorem 1, for any $\epsilon_n = o(\frac{1}{\ln n})$, from (1) (10), $|\alpha_n| = o(\ln n)$ and Fact 2, we obtain (11) here. From $\mathbb{E}[X] = \Omega((\ln n)^3)$ and $\epsilon_n =$

$o(\frac{1}{\ln n})$, it follows that $(1 \pm \epsilon_n)\mathbb{E}[X] = \Omega((\ln n)^3)$, which along with (11) enables the use of Theorem 6 to yield that for $\mathbb{E}[X] = \Omega((\ln n)^3)$ and any $\epsilon_n = o(\frac{1}{\ln n})$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}[G_u(n, P_n, (1 \pm \epsilon_n)\mathbb{E}[X]) \text{ is } k\text{-robust.}] \\ &= \begin{cases} 0, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty, \\ 1, & \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty. \end{cases} \end{aligned} \quad (23)$$

Since k -robustness is a monotone increasing graph property according to [34, Lemma 3], Theorem 4 is proved by (23) and Lemma 2. ■

B. The Proof of Theorem 5

Since k -robustness implies k -connectivity by [27, Lemma 1], the zero law of Theorem 5 is clear from Theorem 2 and Remark 1 in view that under conditions of Theorem 5, if $\lim_{n \rightarrow \infty} \alpha_n = -\infty$,

$$\begin{aligned} & \mathbb{P}[G_b(n, P_n, p_n) \text{ is } k\text{-robust.}] \\ & \leq \mathbb{P}[G_b(n, P_n, p_n) \text{ is } k\text{-connected.}] \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (24)$$

Below we prove the one law of Theorem 5. Note that (5) is the same as (2), and we have condition $|\alpha_n| = o(\ln n)$. Then as proved in Theorem 2, given (2) and $|\alpha_n| = o(\ln n)$, we obtain (16), which together with condition $P_n = \Omega(n(\ln n)^5)$ leads to

$$p_n \sim \sqrt{\frac{\ln n}{nP_n}} = O\left(\sqrt{\frac{\ln n}{n^2(\ln n)^5}}\right) = O\left(\frac{1}{n(\ln n)^2}\right). \quad (25)$$

Noting that (25) implies condition $p_n = O(\frac{1}{n \ln n})$ in Lemma 3, we apply Lemmas 1 and 3, and condition (5) to derive the following: there exists $\hat{p}_n = \frac{\ln n + (k-1) \ln \ln n + \alpha_n - O(1)}{n}$ such that if $\lim_{n \rightarrow \infty} \alpha_n = \infty$,

$$\begin{aligned} & \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ is } k\text{-robust.}] \\ & \geq \mathbb{P}[\text{Graph } G(n, \hat{p}_n) \text{ is } k\text{-robust.}] - o(1) \rightarrow 1, \text{ as } n \rightarrow \infty. \end{aligned} \quad (26)$$

The proof of Theorem 5 is completed via (24) and (26). ■

C. The Proof of Theorem 6

The zero law of Theorem 6 is proved below by an approach similar to that of Theorem 5. Since k -robustness implies k -connectivity by [27, Lemma 1], the zero law of Theorem 6 is clear from Theorem 3 in view that under conditions of Theorem 6, if $\lim_{n \rightarrow \infty} \alpha_n = -\infty$,

$$\begin{aligned} & \mathbb{P}[G_u(n, P_n, K_n) \text{ is } k\text{-robust.}] \\ & \leq \mathbb{P}[G_u(n, P_n, K_n) \text{ is } k\text{-connected.}] \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (27)$$

Below we establish the one law of Theorem 6 with the help of Theorem 5. Given $K_n = \Omega((\ln n)^3) = \omega(\ln n)$, we use Lemma 5 to obtain that with p_n set by

$$p_n = \frac{K_n}{P_n} \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right), \quad (28)$$

it holds that

$$\begin{aligned} & \mathbb{P}[\text{Graph } G_u(n, P_n, K_n) \text{ is } k\text{-robust.}] \\ & \geq \mathbb{P}[\text{Graph } G_b(n, P_n, p_n) \text{ is } k\text{-robust.}] - o(1). \end{aligned} \quad (29)$$

Note that (6) is the same as (3); and $|\alpha_n| = o(\ln n)$ holds as a condition. Then as shown in Theorem 3, from (3), $|\alpha_n| = o(\ln n)$ and Fact 2, we obtain (21) here, which together with $K_n = \Omega((\ln n)^3)$ results in

$$P_n \sim \frac{nK_n^2}{\ln n} = \Omega(n(\ln n)^5), \quad (30)$$

From $K_n = \Omega((\ln n)^3)$ and (28), it follows that

$$\begin{aligned} p_n^2 P_n &= \left[\frac{K_n}{P_n} \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right)\right]^2 \cdot P_n \\ &= \frac{K_n^2}{P_n} \cdot \left[1 - O\left(\frac{1}{\ln n}\right)\right]. \end{aligned} \quad (31)$$

By (6) (31) and Fact 2, it is clear that

$$p_n^2 P_n = \frac{\ln n + (k-1) \ln \ln n + \alpha_n - O(1)}{n}. \quad (32)$$

Given (30) (32) and $|\alpha_n| = o(\ln n)$, we use Theorem 5 and (29) to get that if $\lim_{n \rightarrow \infty} \alpha_n = \infty$,

$$\mathbb{P}[G_u(n, P_n, K_n) \text{ is } k\text{-robust.}] \rightarrow 1, \text{ as } n \rightarrow \infty. \quad (33)$$

The proof of Theorem 6 is completed via (27) and (33). ■

VI. ESTABLISHING LEMMAS IN SECTION III

Lemmas 1 and 4 are clear in Section III. Below we prove Lemmas 2, 3 and 5.

A. The Proof of Lemma 2

According to [2, Lemma 3], for any monotone increasing graph property \mathcal{I} and any $|\epsilon_n| < 1$,

$$\begin{aligned} & \mathbb{P}[G(n, P_n, \mathcal{D}) \text{ has } \mathcal{I}.] - \mathbb{P}[G_u(n, P_n, (1 - \epsilon_n)\mathbb{E}[X]) \text{ has } \mathcal{I}.] \\ & \geq \{1 - \mathbb{P}[X < (1 - \epsilon_n)\mathbb{E}[X]]\}^n - 1, \end{aligned} \quad (34)$$

and

$$\begin{aligned} & \mathbb{P}[G(n, P_n, \mathcal{D}) \text{ has } \mathcal{I}.] - \mathbb{P}[G_u(n, P_n, (1 + \epsilon_n)\mathbb{E}[X]) \text{ has } \mathcal{I}.] \\ & \leq 1 - \{1 - \mathbb{P}[X > (1 + \epsilon_n)\mathbb{E}[X]]\}^n. \end{aligned} \quad (35)$$

By (34) (35) and the fact that $\lim_{n \rightarrow \infty} (1 - m_n)^n = 1$ for $m_n = o(\frac{1}{n})$ (this can be proved by a simple Taylor series expansion as in [7, Fact 2]), the proof of Lemma 2 is completed once we demonstrate that with $\text{Var}[X] = o\left(\frac{\{\mathbb{E}[X]\}^2}{n(\ln n)^2}\right)$, there exists $\epsilon_n = o(\frac{1}{\ln n})$ such that

$$\mathbb{P}[X < (1 - \epsilon_n)\mathbb{E}[X]] = o\left(\frac{1}{n}\right), \quad (36)$$

and

$$\mathbb{P}[X > (1 + \epsilon_n)\mathbb{E}[X]] = o\left(\frac{1}{n}\right). \quad (37)$$

To prove (36) and (37), Chebyshev's inequality yields

$$\mathbb{P}[|X - \mathbb{E}[X]| > \epsilon_n \mathbb{E}[X]] \leq \frac{\text{Var}[X]}{\{\epsilon_n \mathbb{E}[X]\}^2}. \quad (38)$$

We set ϵ_n by $\epsilon_n = \sqrt[4]{\frac{n \text{Var}[X]}{\{\mathbb{E}[X]\}^2}} \cdot \frac{1}{\sqrt{\ln n}}$. Then given condition $\text{Var}[X] = o\left(\frac{\{\mathbb{E}[X]\}^2}{n(\ln n)^2}\right)$, we obtain

$$\epsilon_n = o\left(\sqrt[4]{\frac{1}{(\ln n)^2}}\right) \cdot \frac{1}{\sqrt{\ln n}} = o\left(\frac{1}{\ln n}\right), \quad (39)$$

and

$$\frac{\text{Var}[X]}{\{\epsilon_n \mathbb{E}[X]\}^2} = \sqrt{\frac{\text{Var}[X]}{n\{\mathbb{E}[X]\}^2}} \cdot \ln n = o\left(\frac{1}{n}\right). \quad (40)$$

By (38) (39) and (40), it is straightforward to see that (36) and (37) hold with $\epsilon_n = o\left(\frac{1}{\ln n}\right)$. Therefore, we have completed the proof of Lemma 2. ■

B. The Proof of Lemma 3

In view of [14, Lemma 3], if $p_n^2 P_n < 1$ and $p_n = o\left(\frac{1}{n}\right)$, with $\hat{p}_n := p_n^2 P_n \cdot \left(1 - np_n + 2p_n - \frac{p_n^2 P_n}{2}\right)$, then (9) follows. Given conditions $p_n = O\left(\frac{1}{n \ln n}\right)$ and $p_n^2 P_n = O\left(\frac{1}{\ln n}\right)$ in Lemma 3, $p_n^2 P_n < 1$ and $p_n = o\left(\frac{1}{n}\right)$ clearly hold. Then Lemma 3 is proved once we show \hat{p}_n satisfies $\hat{p}_n = p_n^2 P_n \cdot [1 - O\left(\frac{1}{\ln n}\right)]$, which is easy to see via

$$\begin{aligned} & -np_n + 2p_n - \frac{p_n^2 P_n}{2} \\ &= (-n+2) \cdot O\left(\frac{1}{n \ln n}\right) - \frac{1}{2} \cdot O\left(\frac{1}{\ln n}\right) = -O\left(\frac{1}{\ln n}\right). \end{aligned}$$

Hence, the proof of Lemma 3 is completed. ■

C. The Proof of Lemma 5

We use Lemma 4 to prove Lemma 5. From conditions $K_n = \omega(\ln n)$ and $p_n = \frac{K_n}{P_n} \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right)$, we first obtain $p_n P_n = \omega(\ln n)$ and then for all n sufficiently large,

$$\begin{aligned} & K_n - \left[p_n P_n + \sqrt{3(p_n P_n + \ln n) \ln n}\right] \\ &= K_n \sqrt{\frac{3 \ln n}{K_n}} - \sqrt{3 \left[K_n \left(1 - \sqrt{\frac{3 \ln n}{K_n}}\right) + \ln n\right] \ln n} \\ &= \sqrt{3 K_n \ln n} - \sqrt{3 \left[K_n + \sqrt{\ln n} \left(\sqrt{\ln n} - \sqrt{3 K_n}\right)\right] \ln n} \\ &\geq 0. \end{aligned}$$

Then by Lemma 4, Lemma 5 is now established. ■

VII. NUMERICAL EXPERIMENTS

We present numerical experiments in the non-asymptotic regime to confirm our theoretical results.

Figure 2 depicts the probability that binomial random intersection graph $G_b(n, P, p)$ has k -connectivity or k -robustness, for $k = 2, 6$. Similarly, Figure 3 illustrates the probability of k -connectivity or k -robustness for $k = 3, 4$ in

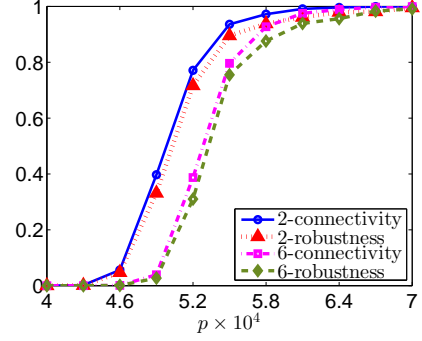


Fig. 2: A plot of the empirical probabilities that binomial random intersection graph $G_b(n, P, p)$ has k -connectivity or k -robustness as a function of p , with $n = 2,000$, $P = 20,000$ and $k = 2, 6$.

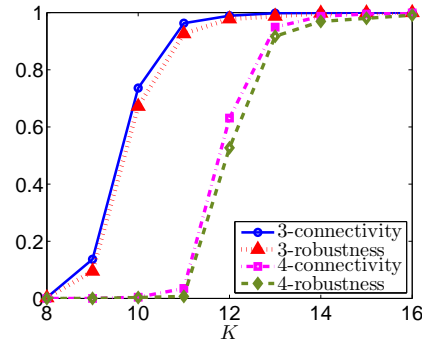


Fig. 3: A plot of the empirical probabilities that uniform random intersection graph $G_u(n, P, K)$ has k -connectivity or k -robustness as a function of K , with $n = 2,000$, $P = 20,000$ and $k = 3, 4$.

uniform random intersection graph $G_u(n, P, K)$. In all set of experiments, we fix the number of nodes at $n = 2,000$ and the object pool size $P = 20,000$. For each pair (n, P, p) (resp., (n, P, K)), we generate 1,000 independent samples of $G_b(n, P, p)$ (resp., $G_u(n, P, K)$) and count the number of times that the obtained graphs are k -connected or k -robust. Then the counts divided by 1,000 become the corresponding empirical probabilities. As illustrated in Figures 2 and 3, there is an evident threshold in the probabilities of k -connectivity and k -robustness. Also, for each k , the curves of k -connectivity and k -robustness are close to each other. These numerical results are in agreement with our analytical findings in the theorems.

VIII. RELATED WORK

For connectivity (i.e., k -connectivity with $k = 1$) in binomial random intersection graph $G_b(n, P_n, p_n)$, Rybarczyk establishes the exact probability [13] and a zero-one law [13], [14]. She further shows a zero-one law for k -connectivity [13], [14]. Our Theorem 2 provides not only a zero-one law, but also the exact probability to deliver a precise understanding of k -connectivity.

For connectivity in uniform random intersection graph $G_u(n, P_n, K_n)$, Rybarczyk [16] derives the exact probability and a zero-one law, while Blackburn and Gerke [15], Yağan

and Makowski [4], and Zhao *et al.* [5], [7] also obtain zero–one laws. Rybarczyk [14] implicitly shows a zero–one law for k -connectivity in $G_u(n, P_n, K_n)$. Our Theorem 3 also gives a zero–one law. In addition, it gives the exact probability to provide an accurate understanding of k -connectivity.

For general random intersection graph $G(n, P_n, \mathcal{D})$, Godehardt and Jaworski [12] investigate its degree distribution and Bloznelis *et al.* [2] explore its component evolution, but provides neither a zero–one law nor the exact probability of its k -connectivity property reported in our work.

To date, there have not been any results reported on the (k) -robustness of random intersection graphs by others. As noted in Lemma 1, Zhang and Sundaram [27] present a zero–one law for k -robustness in an Erdős–Rényi graph.

For random intersection graphs in this paper, two nodes have an edge in between if their object sets share at least one object. A natural variant is to define graphs with edges only between nodes which have at least s objects in common (instead of just 1) for some positive integer s . Zhao *et al.* [22]–[24] consider k -connectivity in graphs under this definition. In addition, (k) -connectivity of other random graphs have also been investigated in the literature [25], [26].

IX. CONCLUSION AND FUTURE WORK

Under a general random intersection graph model, we derive sharp zero–one laws for k -connectivity and k -robustness, as well as the asymptotically exact probability of k -connectivity, where k is an arbitrary positive integer. A future direction is to obtain the asymptotically exact probability of k -robustness for a precise characterization on the robustness strength.

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